

On differentiability with respect to the initial data of a solution of an SDE with Lévy noise and discontinuous coefficients

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Abstract. We construct a stochastic flow generated by an SDE

$$\begin{cases} d\varphi_t(x) = a(\varphi_t(x))dt + dZ(t), & t \geq 0, \\ \varphi_0(x) = x, & x \in \mathbb{R}, \end{cases}$$

where a is a function of bounded variation on \mathbb{R} , $\{Z(t), t \geq 0\}$ is a symmetric stable process with $\alpha \in (1, 2)$. It is proved that this flow is non-coalescing and Sobolev differentiable in x . The representation for the derivative is given.

Introduction

Consider an SDE

$$\varphi_t(x) = x + \int_0^t a(\varphi_s(x))ds + Z(t), t \geq 0, \quad (0.1)$$

where $x \in \mathbb{R}$, a is a bounded measurable function on \mathbb{R} , $\{Z(t), t \geq 0\}$ is a symmetric stable process with the exponent $\alpha \in (1, 2)$, i.e. $\{Z(t), t \geq 0\}$ is a Lévy process with its characteristic function being equal to

$$E \exp\{i\lambda Z(t)\} = \exp\{-ct|\lambda|^\alpha\}, \lambda \in \mathbb{R},$$

where $c > 0$ is a constant.

It was proved in [8] that there exists a unique strong solution of (0.1). The solution $\{\varphi_t(x), t \geq 0\}$ is a strong Markov process (see [12]). Besides, it is continuously dependent on x :

$$\forall T > 0 \forall x_0 \in \mathbb{R} : \sup_{t \in [0, T]} |\varphi_t(x) - \varphi_t(x_0)| \xrightarrow{P} 0, x \rightarrow x_0.$$

In this paper we construct a modification of $\{\varphi_t(x)\}$ which is càdlàg in t and monotonous in x . We prove that if a function a has a locally bounded

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variation, then $\{\varphi_t(x)\}$ is Sobolev differentiable in x a.s. and the derivative has the following representation

$$\nabla \varphi_t(x) = \exp \left\{ \int_{\mathbb{R}} L_t^{\varphi(x)}(y) da(y) \right\}, \quad (0.2)$$

where $L_t^{\varphi(x)}(y)$ is a local time of the process $\{\varphi_s(x), s \in [0, t]\}$ at the point y .

Formula (0.2) can be easily explained for $a \in C^1(\mathbb{R})$. Indeed, in this case for each ω , equation (0.1) can be considered as an integral equation with continuously differentiable coefficients. Then $\varphi_t(x)$ is continuously differentiable in x and $\nabla \varphi_t(x) := \frac{\partial \varphi_t(x)}{\partial x}$ satisfies the linear equation

$$\nabla \varphi_t(x) = 1 + \int_0^t a'(\varphi_s(x)) \nabla \varphi_s(x) ds. \quad (0.3)$$

So

$$\nabla \varphi_t(x) = \exp \left\{ \int_0^t a'(\varphi_s(x)) ds \right\}. \quad (0.4)$$

By the occupation times formula [3], the r.h.s. of (0.4) is equal to

$$\exp \left\{ \int_{\mathbb{R}} a'(y) L_t^{\varphi(x)}(y) dy \right\} = \exp \left\{ \int_{\mathbb{R}} L_t^{\varphi(x)}(y) da(y) \right\} \text{ a.s.}$$

Remark 1. All the technical details needed for the existence of local time such as the validity of occupation times formula, the existence of the integrals etc. will be given in the next sections.

To prove (0.2) for a being a function of bounded variation we will use an approximation of (0.1) by SDEs with C^1 drifts.

If $\{Z(t), t \geq 0\}$ is a Wiener process then the similar problem is well-studied even for non-additive noises (see for example [2, 4, 6, 7]). Note that most techniques used in a Wiener case for non-smooth a (Zvonkin's transformation, Tanaka's formula, Girsanov's formula etc.) are inapplicable to a case of Lévy process.

1 Construction

In this section we construct a version of $\{\varphi_t(x), t \geq 0\}$ satisfying some measurability properties and prove existence of a local time for the process $\varphi_t(x)$.

Put $\mathcal{F}_t = \sigma\{Z(s) : 0 \leq s \leq t\}$.

Proposition 1. *Let $a(x)$, $x \in \mathbb{R}$, be a bounded measurable function. Then*

- 1) *There exists a unique strong solution of (0.1), i.e. a \mathcal{F}_t -adapted càdlàg process $\{\varphi_t(x), t \geq 0\}$ that satisfies (0.1) almost surely.*
- 2) *The process $\{\varphi_t(x), t \geq 0\}$ is continuous w.r.t. x in probability in topology of uniform convergence:*

$$\forall T > 0 \forall x_0 \in \mathbb{R} : \sup_{t \in [0, T]} |\varphi_t(x) - \varphi_t(x_0)| \xrightarrow{P} 0, \quad x \rightarrow x_0. \quad (1.5)$$

- 3) *The process $\{\varphi_t(x), t \geq 0\}$ is a homogeneous strong Markov process. It has a continuous transition density $p_t(x, y)$. Moreover*

$$\forall T > 0 \exists N_T = N_{T, \|a\|_{L_p}} \forall t \in (0, T] \forall x \in \mathbb{R}, y \in \mathbb{R} :$$

$$p_t(x, y) \leq \frac{N_T t}{(t^{1/\alpha} + |y - x|)^{\alpha+1}}, \quad (1.6)$$

where $\alpha \in (1, 2)$ is a parameter of a stable process $\{Z(t), t \geq 0\}$.

- 4) *If $x_1 \leq x_2$, then*

$$P\{\varphi_t(x_1) = \varphi_t(x_2), t \geq \sigma_{x_1, x_2}\} = 1, \quad (1.7)$$

where

$$\sigma_{x_1, x_2} = \inf\{t \geq 0 : \varphi_t(x_1) \geq \varphi_t(x_2)\}.$$

- 5) *The process $\{\varphi_t(x), t \geq 0, x \in \mathbb{R}\}$ can be selected such that*

- a) *it is monotonous in x :*

$$\forall \omega \in \Omega \forall x_1 \leq x_2 \forall t \geq 0 : \varphi_t(x_1, \omega) \leq \varphi_t(x_2, \omega); \quad (1.8)$$

- b) *it is càdlàg in x for any fixed t and ω ;*

- c) *for any $T > 0$ a map*

$$[0, T] \times \mathbb{R} \times \Omega \ni (t, x, \omega) \mapsto \varphi_t(x, \omega)$$

is $\mathcal{B}([0, T]) \times \mathcal{B}(\mathbb{R}) \times \mathcal{F}_T$ -measurable.

Proof. For a proof of 1), 2), see [8], 3) is proved in [9, 12].

Let us prove 4). Since the process

$$\varphi_t(x_2) - \varphi_t(x_1) = \int_0^t (a(\varphi_s(x_2)) - a(\varphi_s(x_1))) ds, \quad t \geq 0,$$

is continuous in t , then

$$\sigma_{x_1, x_2} = \inf\{t \geq 0 : \varphi_t(x_1) = \varphi_t(x_2)\},$$

and it is easy to see that the process

$$\tilde{\varphi}_t(x_1) := \begin{cases} \varphi_t(x_1), & t \leq \sigma_{x_1, x_2}, \\ \varphi_t(x_2), & t > \sigma_{x_1, x_2}, \end{cases}$$

is a solution of (0.1) with initial value x_1 . By the uniqueness of the solution this implies (1.7).

We construct a version of the solution that satisfies properties of the last part of Proposition 1. Let $\tilde{\Omega}$ be a set of full measure such that (0.1) and (1.8) are satisfied for all $\omega \in \tilde{\Omega}$, $t \geq 0$, and rational x . The monotonicity and (1.5) imply that $\forall \omega \in \tilde{\Omega} \forall x_0 \in \mathbb{Q} \forall T > 0$:

$$\sup_{t \in [0, T]} |\varphi_t(x, \omega) - \varphi_t(x_0, \omega)| \rightarrow 0, \quad x \rightarrow x_0, x \in \mathbb{Q}.$$

It is easy to see that

$$\tilde{\varphi}_t(x, \omega) := \begin{cases} \varphi_t(x), & x \in \mathbb{Q} \text{ and } \omega \in \tilde{\Omega}, \\ \lim_{y \downarrow x, y \in \mathbb{Q}} \varphi_t(y), & x \notin \mathbb{Q} \text{ and } \omega \in \tilde{\Omega}, \\ x, & \omega \notin \tilde{\Omega}, \end{cases}$$

is a version of $\{\varphi_t(x), t \geq 0, x \in \mathbb{R}\}$ that satisfies 5).

The Proposition is proved. □

Remark 2. Later on we will always consider a version of $\{\varphi_t(x)\}$ that satisfies 5).

There are a few different approaches to the notion of a local time. We consider a local time as a density of the occupation measure. Recall the definition and some properties of local times from [3, 5].

Let $X(t), t \geq 0$, be a measurable function. Define a measure

$$\nu_t(A) := \lambda\{s : X(s) \in A, s \in [0, t]\},$$

where λ is a Lebesgue measure.

If the measure ν_t is absolutely continuous w.r.t. λ , then its Radon–Nikodym derivative $\alpha(y, t) = \frac{d\nu_t(y)}{d\lambda}$ is called a local time of X relative to $[0, t]$. In particular,

$$\begin{aligned} \nu_t(A) &= \int_A \alpha_t(y) dy, \quad A \in \mathcal{B}(\mathbb{R}), \\ \int_0^t f(X(s)) ds &= \int_{\mathbb{R}} f(y) \nu_t(dy) = \int_{\mathbb{R}} f(y) \alpha_t(y) dy, \end{aligned} \quad (1.9)$$

where f is a measurable function for which at least one integral in (1.9) make sense.

Let $X(t), t \geq 0$, be a measurable stochastic process. We say that the local time of X exists a.s. if almost all trajectories have a local time.

Assume that for any $t_1 < t_2 < \dots < t_n$ the distribution of $(X(t_1), \dots, X(t_n))$ is absolutely continuous. Let $p(x_1, \dots, x_n, t_1, \dots, t_n)$ be the corresponding density.

Put

$$q_t(x_1, \dots, x_n) = \int_0^t \dots \int_0^t p(x_1, \dots, x_n, t_1, \dots, t_n) dt_1 \dots dt_n.$$

Theorem 1. *If for some $n \geq 2$ the function q_t is continuous, then the local time exists a.s., and*

$$E\alpha_t(x_1) \cdot \dots \cdot \alpha_t(x_n) = q_t(x_1, \dots, x_n). \quad (1.10)$$

Proof see in [3].

Remark 3. It was mentioned in the proof (see also [5], Sect.25) that

$$(2\varepsilon)^{-1} \int_0^t \mathbf{1}_{\{|X(s)-y|<\varepsilon\}} ds \rightarrow \alpha_t(y), \quad \varepsilon \rightarrow 0+, \quad (1.11)$$

in L_2 -sense for any $y \in \mathbb{R}$, and almost surely for λ -a.a. y . It follows from (1.11) and standard results on existence of measurable version of a limit (see, for example, [14]), that the local time can be selected measurable in (y, ω) . Only a such modification will be considered further.

Remark 4. It is sufficient to assume in Theorem 1 that $(X(t_1), \dots, X(t_n))$ is absolutely continuous for λ^n -a.a. $(t_1, \dots, t_n) \in [0, t]^n$.

Remark 5. Note that if the local time exists a.s., then (1.9) is satisfied with probability one for any measurable non-negative function f . The exceptional set is independent of f .

Let us return to equation (0.1).

Proposition 2. *There exists a process $\alpha_x(y, t)$, $x \in \mathbb{R}, y \in \mathbb{R}, t \geq 0$, such that*

- 1) *for any fixed x : $\alpha_x(y, t)$ is a local time of $\varphi_s(x)$, $s \in [0, t]$;*
- 2) *$\alpha_x(y, t)$ is measurable in (x, y, ω, t) ;*
- 3) *for any $x \in \mathbb{R}, t > 0$, a map $y \mapsto \alpha_x(y, t)$ is continuous in L_2 .*

Proof. The existence of the local time for fixed x follows from Theorem 1 and Proposition 1. Indeed, let $n = 2$, then

$$\begin{aligned} p(x_1, x_2, t_1, t_2) &= p_{t_1}(x, x_1) p_{t_2 - t_1}(x_1, x_2) \leq \\ &\leq (N_t)^2 t_1^{1 - \frac{\alpha+1}{\alpha}} (t_2 - t_1)^{1 - \frac{\alpha+1}{\alpha}} = \\ &= (N_t)^2 t_1^{-\frac{1}{\alpha}} (t_2 - t_1)^{-\frac{1}{\alpha}}, 0 < t_1 < t_2 \leq t. \end{aligned}$$

Observe that

$$\begin{aligned} &\int_0^t \int_0^{t_2} (N_t)^2 t_1^{-\frac{1}{\alpha}} (t_2 - t_1)^{-\frac{1}{\alpha}} dt_1 dt_2 = \\ &= (N_t)^2 \int_0^t \int_0^1 (zt_2)^{-\frac{1}{\alpha}} (t_2 - zt_2)^{-\frac{1}{\alpha}} t_2 dz dt_2 = \\ &= (N_t)^2 \int_0^t t_2^{1 - \frac{2}{\alpha}} B\left(1 - \frac{1}{\alpha}, 1 - \frac{1}{\alpha}\right) dt_2 < \infty. \end{aligned} \tag{1.12}$$

Above we have used that $0 < 1 - \frac{1}{\alpha}$ and $1 - \frac{2}{\alpha} > -1$ because $\alpha \in (1, 2)$.

The continuity of q_t follows from the Lebesgue dominated convergence theorem.

Remark 3 allows us to select a measurable in (x, y, ω, t) modification.

The Proposition is proved. \square

Remark 6. The local time from Proposition 2 coincides with that obtained by N.I.Portenko [10], who considered it as a W -functional from Markov process.

In the following statement we obtain the exponential integrability of the local time.

Proposition 3. *For any $t > 0, \mu > 0$, there exists $c = c(t, \|a\|_\infty, \mu)$ such that*

$$\forall x, y \in \mathbb{R} \quad E \exp\{\mu \alpha_x(y, t)\} \leq c. \quad (1.13)$$

A possible way to prove (1.13) is to expand the exponent in a Taylor series, then to use estimate (1.6), and to make calculations similar to (1.12) and formula (1.10). However it is easier to apply the following result of N.I.Portenko.

Lemma 1. *Assume that $\{\beta(t), t \in [0, T]\}$ is a non-negative measurable process adapted to a flow $\{\mathcal{F}_t, t \in [0, T]\}$. Assume that for $0 \leq s \leq t \leq T$*

$$E \left\{ \int_s^t \beta(\tau) d\tau / \mathcal{F}_s \right\} \leq \rho(s, t),$$

where $\rho(s, t)$ is a non-random integral function satisfying the following conditions

- a) $\rho(t_1, t_2) \leq \rho(t_3, t_4)$ if $(t_1, t_2) \subset (t_3, t_4)$;
- b) $\lim_{h \downarrow 0} \sup_{0 \leq s \leq t \leq h} \rho(s, t) = 0$.

Then for any λ

$$E \exp \left\{ \lambda \int_0^T \beta(\tau) d\tau \right\} \leq c,$$

where c depends only on λ, T and ρ .

See [13], Lemma 1.1 or [11], Lemma 1 for the proof.

Let now $\beta_\varepsilon(t) = (2\varepsilon)^{-1} \mathbf{1}_{|\varphi_t(x) - y| < \varepsilon}$. Similarly to (1.12) we obtain that uniformly in x, y, ε

$$\begin{aligned} & E \left(\int_s^t (2\varepsilon)^{-1} \mathbf{1}_{|\varphi_\tau(x) - y| < \varepsilon} d\tau / \mathcal{F}_s \right) = \\ &= \int_0^{t-s} \int_{y-\varepsilon}^{y+\varepsilon} (2\varepsilon)^{-1} p_r(\varphi_s(x), u) du dr \leq \\ &\leq \int_0^{t-s} \frac{N_T r}{r^{\frac{\alpha+1}{\alpha}}} dr = N_T (t-s)^{\frac{\alpha-1}{\alpha}} \cdot \left(\frac{\alpha}{\alpha-1} \right) =: \rho(s, t), \quad 0 \leq s \leq t \leq T. \end{aligned}$$

This implies the uniform in x, y, ε estimate of

$$E \exp \left\{ \lambda \int_0^t (2\varepsilon)^{-1} \mathbf{1}_{|\varphi_s(x) - y| < \varepsilon} ds \right\}.$$

To conclude the proof, it remains to make $\varepsilon \rightarrow 0$ and apply Fatou's lemma .
Proposition 3 is proved.

2 Representation of the derivative

Consider equation (0.1). Assume that a is continuously differentiable. Then for each $\omega \in \Omega$, (0.1) can be considered as an integral equation with C^1 -coefficients. So $\varphi_t(x)$ is differentiable in x and $\nabla \varphi_t(x) = \frac{\partial \varphi_t(x)}{\partial x}$ satisfies a linear equation

$$\nabla \varphi_t(x) = 1 + \int_0^t a'(\varphi_s(x)) \nabla \varphi_s(x) ds.$$

Thus

$$\nabla \varphi_t(x) = \exp \left\{ \int_0^t a'(\varphi_s(x)) ds \right\}.$$

Applying (1.9) and Proposition 2 we get

$$\nabla \varphi_t(x) = \exp \left\{ \int_{\mathbb{R}} a'(y) \alpha_x(y, t) dy \right\} = \exp \left\{ \int_{\mathbb{R}} \alpha_x(y, t) da(y) \right\} \text{ a.s.}$$

Note that generally speaking the exceptional set depends on x and t . By Fubini's theorem

$$P \left\{ \nabla \varphi_t(x) = \exp \left\{ \int_{\mathbb{R}} \alpha_x(y, t) da(y) \right\} \text{ for } \lambda\text{-a.a. } x \right\} = 1. \quad (2.14)$$

We will justify representation (2.14) for solution of (0.1), where a function a is not necessarily C^1 , but it has a finite variation. We need some definitions and facts on Sobolev spaces.

Definition 1. A function $f : [a, b] \rightarrow \mathbb{R}$ belongs to a Sobolev space $W_p^1([a, b])$, $p \geq 1$, if f has an absolutely continuous modification and $\frac{df}{dx} \in L_p([a, b])$.

Put

$$\|f\|_{p,1} := \|f\|_{W_p^1([a,b])} := \|f\|_{L_p([a,b])} + \left\| \frac{df}{dx} \right\|_{L_p([a,b])}.$$

It is well known that $(W_p^1([a, b]), \|\cdot\|_{p,1})$ is a Banach space. So if $\{f_n\} \subset W_p^1([a, b])$ is such that

$$f_n \rightarrow f, \text{ and } \frac{df_n}{dx} \rightarrow g \text{ as } n \rightarrow \infty \text{ in } L_p, \quad (2.15)$$

then

$$f \in W_p^1([a, b]), \quad g = \frac{df}{dx}. \quad (2.16)$$

Definition 2. A measurable function $f : \mathbb{R} \rightarrow \mathbb{R}$ belongs to the space $W_{p,loc}^1$ if its restriction to any segment $[a, b]$ lies in $W_p^1([a, b])$.

The main result of the paper is the following theorem.

Theorem 2. Assume that $a(x)$, $x \in \mathbb{R}$, is a measurable bounded function and its restriction to any interval has a finite variation. Then $\varphi_t(\cdot) \in W_{p,loc}^1$ for any $p \geq 1$ a.s. and representation (2.14) holds true for any $t > 0$.

Corollary 1. For all $\{x_1, x_2\} \subset \mathbb{R}$, $x_1 \neq x_2$,

$$P \{ \varphi_t(x_1) \neq \varphi_t(x_2), \quad t \geq 0 \} = 1.$$

This result can be obtained similarly to that of [1], Theorem 1.

Proof of Theorem 2. Assume at first that a is a function of bounded variation. Set

$$a_n(x) := \int_{\mathbb{R}} a(y) g_n(x - y) dy,$$

where

$$g_n(x) = ng(nx), \quad g \in C_0^\infty(\mathbb{R}), \quad g \geq 0, \quad \text{and} \quad \int_{\mathbb{R}} g(z) dz = 1.$$

Then

$$\sup_{n,x} |a_n(x)| \leq \|a\|_\infty = \sup_x |a(x)|,$$

$a_n \in C^\infty(\mathbb{R})$, $a_n(x) \rightarrow a(x)$, $n \rightarrow \infty$, for all points of continuity of a , and

$$\text{Var}(a_n) \leq \text{Var}(a). \quad (2.17)$$

Let $\varphi_t^n(x)$ be a solution of (0.1) with a_n instead of a , $\alpha_x^n(y, t)$ be its local time. Then

$$P \left\{ \nabla \varphi_t^n(x) = \exp \left\{ \int_{\mathbb{R}} \alpha_x^n(y, t) da_n(y) \right\} \text{ for } \lambda\text{-a.a } x \right\} = 1.$$

It follows from [8] that

$$\forall x \quad \forall T \geq 0 : \quad \sup_{t \in [0, T]} |\varphi_t^n(x) - \varphi_t(x)| \xrightarrow{P} 0, \quad n \rightarrow \infty.$$

The uniform boundedness of $\{a_n\}$ implies

$$\forall p \geq 1 : \sup_x \sup_{t \in [0, T]} E |\varphi_t^n(x) - \varphi_t(x)|^p < \infty.$$

So

$$E \int_a^b |\varphi_t^n(x) - \varphi_t(x)|^p dx \rightarrow 0, \quad n \rightarrow \infty$$

for any $t > 0, a \leq b$.

Prove that $\forall p \geq 0 \quad \forall a \leq b \quad \forall t > 0 :$

$$E \int_a^b \left| \exp \left\{ \int_{\mathbb{R}} \alpha_x^n(y, t) da_n(y) \right\} - \exp \left\{ \int_{\mathbb{R}} \alpha_x(y, t) da(y) \right\} \right|^p dx \rightarrow 0, \quad n \rightarrow \infty,$$

By Proposition 3, (2.17), and Jensen's inequality, it suffices to check the convergence

$$\forall x, t : \int_{\mathbb{R}} \alpha_x^n(y, t) da_n(y) \rightarrow \int_{\mathbb{R}} \alpha_x(y, t) da(y), \quad n \rightarrow \infty,$$

in probability or in L_2 -sense.

Assume at first that the function a has a finite support. Let R be such that $\text{supp } a \subset [-R, R], \text{ supp } a_n \subset [-R, R]$.

For simplicity denote a by a_0 and α by α^0 .

Let $-R = y_0 < y_1 < \dots < y_m = R$ be a dissection of $[-R, R]$. Then

$$\begin{aligned} E \left| \int_{\mathbb{R}} \alpha_x^n(y, t) da_n(y) - \int_{\mathbb{R}} \alpha_x(y, t) da(y) \right| &= E \left| \int_{-R}^R \dots - \int_{-R}^R \dots \right| \leq \\ &\leq \sum_{0 \leq j < m} E \left| \int_{y_j}^{y_{j+1}} \left(\alpha_x^n(y, t) - \frac{\int_{y_j}^{y_{j+1}} \alpha_x^n(z, t) dz}{\Delta y_j} \right) da_n(y) \right| + \\ &+ \sum_{0 \leq j < m} E \left| \int_{y_j}^{y_{j+1}} \left(\frac{\int_{y_j}^{y_{j+1}} (\alpha_x^n(z, t) - \alpha_x(z, t)) dz}{\Delta y_j} \right) da_n(y) \right| + \\ &+ \sum_{0 \leq j < m} E \left| \int_{y_j}^{y_{j+1}} \left(\frac{\int_{y_j}^{y_{j+1}} \alpha_x(z, t) dz}{\Delta y_j} \right) (da_n(y) - da(y)) \right| + \\ &+ \sum_{0 \leq j < m} E \left| \int_{y_j}^{y_{j+1}} \left(\frac{\int_{y_j}^{y_{j+1}} \alpha_x(z, t) dz}{\Delta y_j} - \alpha_x(y, t) \right) da(y) \right| \leq \end{aligned}$$

$$\begin{aligned}
&\leq 2 \sup_{l \geq 0} \max_{0 \leq j < m} \sup_{y \in [y_j, y_{j+1}]} E \left| \alpha_x^l(y, t) - \frac{\int_{y_j}^{y_{j+1}} \alpha_x^l(z, t) dz}{\Delta y_j} \right| \text{Var } a_l + \\
&+ \max_{0 \leq j < m} E \left| \int_{y_j}^{y_{j+1}} \alpha_x^n(z, t) dz - \int_{y_j}^{y_{j+1}} \alpha_x(z, t) dz \right| \frac{\text{Var } a_n}{\min_{0 \leq j < m-1} \Delta y_j} + \\
&+ \max_{0 \leq j < m} E \frac{\int_{y_j}^{y_{j+1}} \alpha_x(z, t) dz}{\Delta y_j} \sum_{k=0}^{m-1} |\Delta_k a_n - \Delta_k a_0| = I_1 + I_2 + I_3, \quad (2.18)
\end{aligned}$$

where $\Delta y_j = y_{j+1} - y_j$; $\Delta_k a_n = (a_n(y_{k+1}) - a_n(y_k))$.

Estimate each term in the r.h.s. of (2.18).

By (1.10) and (1.6), for any fixed $t \geq 0$, $x \in \mathbb{R}$, the processes $\alpha_x^l(y, t)$, $y \in [-R, R]$, are equicontinuous in L_2 (and so in L_1) uniformly in $l \geq 0$, i.e.

$\forall \varepsilon > 0 \forall l \geq 0 \exists \delta_1 = \delta_1(\varepsilon) > 0 \forall \{y', y''\} \subset [-R, R], |y' - y''| < \delta :$

$$\sqrt{E(\alpha_x^l(t, y') - \alpha_x^l(t, y''))^2} < \varepsilon.$$

Hence, if $\max_{0 \leq j < m} |y_{j+1} - y_j| < \delta_1$, where $\delta_1 = \delta_1(\varepsilon / (6 \sup_{l \geq 0} \text{Var } a_l))$ then the term I_1 in (2.18) is less than $\frac{\varepsilon}{3}$.

Consider I_2 . By the definition of the local time (see 1.9):

$$\int_{y_j}^{y_{j+1}} \alpha_x^n(z, t) dz = \int_0^t \mathbb{1}_{\varphi_z^n(x) \in [y_j, y_{j+1}]} dz \text{ a.s.}$$

Therefore,

$$\begin{aligned}
E \left| \int_{y_j}^{y_{j+1}} \alpha_x^n(z, t) dz - \int_{y_j}^{y_{j+1}} \alpha_x(z, t) dz \right| \\
\leq E \int_0^t \left| \mathbb{1}_{\varphi_z^n(x) \in [y_j, y_{j+1}]} - \mathbb{1}_{\varphi_z(x) \in [y_j, y_{j+1}]} \right| dz.
\end{aligned}$$

Taking into account that by [8],

$$\sup_{z \in [0, t]} |\varphi_z^n(x) - \varphi_z(x)| \xrightarrow{P} 0, \quad n \rightarrow \infty,$$

we get

$$\mathbb{1}_{\varphi_z(x) \notin [y_j, y_{j+1}]} \left(\mathbb{1}_{\varphi_z^n(x) \in [y_j, y_{j+1}]} - \mathbb{1}_{\varphi_z(x) \in [y_j, y_{j+1}]} \right) \xrightarrow{P} 0, \quad n \rightarrow \infty,$$

for any z .

Since

$$E \int_0^t \mathbb{1}_{\varphi_z(x) \in \{y_j, y_{j+1}\}} dz = \int_{\{y_j\} \cup \{y_{j+1}\}} E \alpha_x(z, t) dz = 0,$$

we have the convergence

$$(\mathbb{1}_{\varphi_z^n(x) \in [y_j, y_{j+1}]} - \mathbb{1}_{\varphi_z(x) \in [y_j, y_{j+1}]}) \xrightarrow{P} 0, \quad n \rightarrow \infty,$$

and consequently the convergence

$$E \left| \int_{y_j}^{y_{j+1}} \alpha_x^n(z, t) dz - \int_{y_j}^{y_{j+1}} \alpha_x(z, t) dz \right| \rightarrow 0, \quad n \rightarrow \infty. \quad (2.19)$$

Recall that

$$a_n(y) \rightarrow a_0(y), \quad n \rightarrow \infty, \quad (2.20)$$

if y is a point of continuity of a_0 . Let us select a dissection $\{y_k\}$ such that all $\{y_k\}$ are points of continuity of a_0 , and $\max_j \Delta y_j < \delta_1$. Use (2.19) and (2.20) and select n_0 such that for any $n \geq n_0$:

$$\frac{\sup_{p \geq 0} \text{Var } a_p}{\min_{0 \leq j < m} \Delta y_j} \cdot \max_{0 \leq j < m} E \left| \int_{y_j}^{y_{j+1}} \alpha_x^n(z, t) dz - \int_{y_j}^{y_{j+1}} \alpha_x(z, t) dz \right| < \frac{\varepsilon}{3}$$

and

$$\sup_{-R \leq z \leq R} E \alpha_x(z, t) \cdot \sum_{k=0}^{m-1} |\Delta_k a_n - \Delta_k a_0| < \frac{\varepsilon}{3}.$$

So the r.h.s. of (2.18) is less than ε and the theorem is proved for finite a .

Let now a be an arbitrary function that satisfies conditions of Theorem 2.

Let $g \in C_0^\infty(\mathbb{R})$; $g(x) = 1$, $|x| \leq 1$. Put $g_n(x) = g(x/n)$, $a_n(x) = g_n(x)a(x)$. Let $\varphi_t^n(x)$ be a solution of (0.1) with a drift coefficient equal to a_n .

Observe that by uniqueness of the solution we have the equality

$$\varphi_t^n(x) = \varphi_t(x) \quad (2.21)$$

for a.a. ω from the event $\{\sup_{z \in [0, t]} |\varphi_z(x)| \leq n\}$.

Let $[c, d]$ be an arbitrary interval. Denote $n_0 = n_0(\omega) = \max_{z \in [0, t]} (|\varphi_z(c)| + |\varphi_z(d)|)$. Making use of (2.21), Proposition 1, and Fubini's theorem we obtain

the equality $\varphi_t(x) = \varphi_t^n(x)$ valid for all $n \geq n_0$, a.a. ω , and λ -a.a. $x \in [c, d]$. Since a_n is finite, $\varphi_t^n(\cdot) \in W_p^1([c, d])$ a.s. Thus $\varphi_t(\cdot) \in W_p^1([c, d])$ a.s. and its derivatives coincide a.s. with that of φ_t^n if $n \geq n_0$. The definition of the local time entails that $\alpha_x^n(y, t) = \alpha_x(y, t)$, $n \geq n_0$, for λ -a.a. $y \in [c, d]$ with probability 1. So formula (2.14) holds true.

Theorem 2 is proved. □

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